This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted. Proofs or counterexamples are required for all problems. Time allowed: 2 hours 30 minutes.

Notation: $\mathbb{R}$ is the real line; $\mathbb{R}^d$ is the d-dimensional Euclidean space; $\mathbb{N}$ is the set of positive integers.

1. Prove that the function $f(x) = (1 + x^3)e^{-|x|}$ is uniformly continuous on $\mathbb{R}$.

2. (a) Give the definition of what it means for a set $A \subset \mathbb{R}^d$ to be connected.

   (b) Let $B \subset \mathbb{R}$ be a bounded non-empty connected set. Prove that B is an interval (i.e., a set of the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$ for some $a, b \in \mathbb{R}$ such that $a \le b$).

3. **Definition.** A function $h\colon A \to B$ is called surjective if for every $y \in B$ there exists $x \in A$ such that $h(x) = y$.

   Let $f, g\colon [0, 1] \to [0, 1]$ be two continuous surjective functions. Prove that there exists $x_0 \in [0, 1]$ such that $f(g(x_0)) = g(f(x_0))$.

4. **Definition.** For $\alpha > 0$ and an interval I, a function $h\colon I \to \mathbb{R}$ is said to satisfy the α-Hölder condition on I if there exists $C > 0$ such that

$$|h(x) - h(y)| \le C|x - y|^\alpha, \quad x, y \in I.$$

   (a) Prove or give a counterexample to the following statement: If f satisfies the $\frac{1}{2}$-Hölder condition on $(-1, 1)$, then f is differentiable on $(-1, 1)$.

   (b) Prove that if g satisfies the 2-Hölder condition on $(-1, 1)$, then g is constant on $(-1, 1)$.

5. Let $\{a_n\}_{n=2}^\infty$ be a sequence of real numbers such that $|a_{n+1} - a_n| \le \frac{|a_n|}{n^2}$ for $n \ge 2$.

   (a) Prove that the sequence $\{a_n\}_{n=2}^\infty$ is bounded. (Hint: first, prove that $|a_n| \le C(1-1/n)$ for an appropriate constant C.)

   (b) Prove that the sequence $\{a_n\}_{n=2}^\infty$ is convergent. (Hint: first, show that the sequence $\{a_n\}$ is Cauchy.)

6. Let f be a non-negative Riemann integrable function on the interval $[0, 1]$. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} n \left(\int_{(k-1)/n}^{k/n} f(x)\, dx \right)^2 = \int_0^1 \big(f(x)\big)^2\, dx.$$