PRELIMINARY EXAM IN ALGEBRA - AUGUST 2026

Time allowed : 2 hours and 30 minutes
No calculators, cell phones, or other electronic devices allowed

1. Show that $\mathbb{Z}[x]/(2x+1)$ is isomorphic to the ring of dyadic rationals, that is, the subring
$$\{x2^{-n} : x, n \in \mathbb{Z}, n \geq 0\}$$
of $\mathbb{Q}$.

2. Prove that a finite field extension E/F is finitely generated. In other words, $E = F(\alpha_1, ..., \alpha_k)$ for some finite k.

3. Prove that a finite field extension E/F is separable if and only if it has exactly $[E : F]$ embeddings into the algebraic closure of F.

4. Let $n \geq 1$ be an integer, and p be a 3 (mod 4) prime number.
(a) Prove that $x^2 + 1$ is irreducible over $\mathbb{F}_p$.
(b) Prove that $(x^2 + 1)^n - p$ is irreducible over $\mathbb{Z}$.

5. Let $n \geq 3$, and ζ_{2^n} be a primitive 2^n-th root of unity.
(a) How many field extensions $K/\mathbb{Q}$ are there such that $[\mathbb{Q}(\zeta_{2^n}) : K] = 2$?
(b) For each extension found in (a), can you write down its generator(s) over $\mathbb{Q}$?
(Hint: For both parts, it may be helpful to recall the following facts: $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^n})/\mathbb{Q}) \cong (\mathbb{Z}/2^n\mathbb{Z})^*$, and $(\mathbb{Z}/2^n\mathbb{Z})^* \cong \mathbb{Z}/2^{n-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.)