Linear algebra qualification exam

August 18, 2026

In the problems below, V and W denote vector spaces, and all vector spaces are finite dimensional over a field $F = \mathbb{R}$ or $\mathbb{C}$ unless specifically mentioned.

1. For vectors $v_1, v_2, v_3, v_4 \in V$ ($\dim V \geq 4$), assume that the vectors $v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4$ are linearly independent. Show that v_1, v_2, v_3, v_4 are also linearly independent.

2. Let $T \in \mathcal{L}(V)$. Suppose that every susbpace of V is an invariant subspace of T. Show that $T = cI$ for some scalar c.

3. Let $S, T \in \mathcal{L}(V, W)$. Suppose T is injective, and $\dim \operatorname{im} S = 3$.

   Show that $\dim \operatorname{null}(S + T) \leq 3$.

4. A *quadratic form* Q on $\mathbb{R}^n$ is a degree 2 homogeneous multivariate polynomial, i.e.

$$Q(x) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

   for some $a_{ij} \in \mathbb{R}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In addition, we say Q is *positive definite* if $Q(x) > 0$ for all $x \neq 0$.

   (a) Show that $Q(x) = x^{\mathrm{tr}} A x$ for some symmetric real $n \times n$ matrix A, and that Q is positive definite if and only if the operator $x \mapsto Ax$ on $\mathbb{R}^n$ is positive.

   (b) Suppose $Q(x) = x^{\mathrm{tr}} A x$ is positive definite. Prove that $\frac{1}{2}(Q(x+y) - Q(x) - Q(y))$ defines an inner product on $\mathbb{R}^n$.

   (c) Show that there exists an $n \times n$ matrix B such that $Q(Bx) = b_1 x_1^2 + \ldots + b_n x_n^2$ for some $b_1, \ldots, b_n \in \mathbb{R}$.

5. Suppose $U \subseteq V$ is a subspace. Recall the notion of the quotient space

$$V/U = \{v + U : v \in V\}$$

with the natural projection $\pi : V \to V/U$ by $v \mapsto v + U$.

In the problems below, suppose that $T \in \mathcal{L}(V)$ and U is invariant under T.

(a) Show that there exists a unique map $\tilde{T} \in \mathcal{L}(V/U)$ such that $\pi \circ T = \tilde{T} \circ \pi$.

(b) Suppose in addition that T is diagonalizable. Show that $\tilde{T}$ is diagonalizable.