Ordinary Differential Equation Preliminary Exam

August 20, 2026

Problem 1 Show that all solutions of

$$x' = x(3 - 3x - y)$$

$$y' = y(3 - x - 3y)$$

with initial conditions satisfying $x(0) > 0$, $y(0) > 0$ must tend to a rest point as $t \to \infty$.

Problem 2. Consider the system

$$x' = -y + x(1 - 3x^2 - 2y^2)$$

$$y' = x + y(1 - 3x^2 - 2y^2).$$

(a) Find the equilibrium points and determine their stability.

(b) Rewrite the equations in polar coordinates.

(c) Show that there exists a closed orbit in some annulus

$$A = \{(x, y) : r_0 \leq \sqrt{x^2 + y^2} \leq R_0\}.$$

Problem 3 Consider the nonlinear system

$$\dot{x} = y, \quad \dot{y} = -x + (1 - x^2)y.$$

(i) Linearize the system at the equilibrium point and classify the stability of the equilibrium.

(ii) Define the energy function

$$V(x, y) = \frac{1}{2}(x^2 + y^2),$$

and compute $\dot{V}$ along solutions.

(iii) Use the sign of $\dot{V}$ to show that trajectories are repelled from a neighborhood of the origin and attracted from sufficiently far away.

Problem 4 Consider the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(i) Find the eigenvalues of A and determine their algebraic multiplicities.

(ii) Determine the geometric multiplicity of the eigenvalue $\lambda = 2$. Explain why A is not diagonalizable.

(iii) Compute the matrix exponential e^{At}.

(iv) Discuss the asymptotic behavior of solutions as $t \to \infty$.