1. Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two sequences of random variables such that X_n and Y_n converge in distribution to X and Y as $n \to \infty$, respectively.

   (a) Suppose $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are defined on the same probability space, and in addition the two sequences are independent. Does $X_n + Y_n$ converge in distribution as $n \to \infty$? If so, describe the limiting distribution.

   Hint: X and Y are not necessarily defined on the same probability space; you might be tempted to write $X + Y$ but if so you need to first explain what it means.

   (b) Suppose $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are defined on the same probability space, but not necessarily independent. Is it true that $X_n + Y_n$ still converges in distribution as $n \to \infty$, and the limiting distribution remains the same as the one in the first part? If your answer is yes, provide a proof. If your answer is no, provide a counter example.

2. Let $(X_n)_{n\geq 1}$ be a sequence of independent geometric random variables with parameter $p \in (0, 1)$. That is, $\mathbb{P}(X_n = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. Find a constant β such that, for all $\varepsilon > 0$,

$$\mathbb{P}(X_n > (\beta + \varepsilon) \ln n \quad \text{i.o.}) = 0,$$

   and

$$\mathbb{P}(X_n > (\beta - \varepsilon) \ln n \quad \text{i.o.}) = 1.$$

   Your value of β may depend on p. Here "i.o." is an abbreviation for "infinitely often" which means that an event occurs infinitely many times in a sequence.

3. Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with probability density function $f(x) = xe^{-x}$ for $x > 0$ and $f(x) = 0$ if $x \leq 0$.

   (a) Compute the tail probability $\mathbb{P}(X_1 > x)$ for $x > 0$.

   (b) Let $b_1 = 0$ and $b_n = \ln n + \ln \ln n$ for $n \geq 2$. Consider the random variables

$$M_n = \max_{1\leq i\leq n} X_i.$$

   Show that $M_n - b_n$ converges in distribution to a nondegenerate random variable. Identify the cumulative distribution function of the limiting random variable.

4. Let $(X_n)_{n=1}^{\infty}$ be independent random variables with

$$\mathbb{P}\left(X_n = \sqrt{n}\right) = \mathbb{P}\left(X_n = -\sqrt{n}\right) = \frac{1}{2\sqrt{n}}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{\sqrt{n}}.$$

   Let $M_n = X_1 + \ldots + X_n$. Find a_n, b_n such that

$$\frac{M_n - a_n}{b_n} \Rightarrow \mathcal{N}(0, 1).$$

   Here $\mathcal{N}(0, 1)$ denotes the standard normal distribution.

   (a) Provide a complete statement of the central limit theorem you choose to apply here.

   (b) Provide the details on how the conditions (in the central limit theorem you choose to apply) are satisfied with the $(a_n)_{n=1}^{\infty}$and $(b_n)_{n=1}^{\infty}$ that you find.
   Hint: You may use the fact that $\sum_{k=1}^{n} k^{\alpha} \sim \int_1^n x^{\alpha} dx$ for all $\alpha \geq -1$. Here $A_n \sim B_n$ means $\lim_{n\to\infty} A_n/B_n = 1$.